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Series analysis of the q-state checkerboard Potts model

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Abstract. The low-temperature expansion of the checkerboard Potts model in a magnetic field is obtained up to order twelve from the so-called disorder solutions, various expansions and results known in the literature. It is shown that this expansion drastically simplifies on the dual of the disorder variety. The low-temperature expansion of the magnetisation is seen to become equal to one (up to order twelve) when it is restricted to the dual of the disorder variety. These results have to be seen as exact formal constraints on the analytic continuation of the low-temperature expansion of the partition function per site. Similar simplifications occur for the susceptibility and higher derivatives with respect to the magnetic field. These expansions are also analysed in the vicinity of these particular varieties.

1. Introduction

Very few anisotropic low- or high-temperature expansions have been obtained for models in statistical mechanics on lattices. One apparent reason for this fact is that, as far as we are concerned with critical behaviour of these models, universality strongly supports the idea that no difference exists between isotropic and anisotropic models. On the other hand, combinatorics is much more involved for anisotropic models than for isotropic ones. However, recent developments in exactly solvable models have emphasised the important role played by anisotropic models (Baxter 1982). Moreover expansions in the vicinity of the disorder solutions have been seen to exhibit many remarkable features (rational or algebraic expressions, etc) (Georges *et al* 1986a, b).

We will focus in this paper on the expansion of the checkerboard Potts (or Ising) model (eventually in a magnetic field) because it is an important model in two-dimensional statistical mechanics for which a great number of exact results or expansions (even anisotropic one) have been accumulated. By taking some appropriate limit one can recover from the expansion of the checkerboard Ising or Potts models the expansions of the triangular, honeycomb, square and anisotropic square models that are already known in the literature up to different orders (Utiyama 1951, Kihara *et al* 1954, Straley and Fisher 1973, Sykes *et al* 1973).

Section 2 of the paper synthesises most of the information on expansions already known in the literature to obtain the low-temperature expansion of the checkerboard

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Potts model in a magnetic field. We also systematically use the constraints that stem from all the exact results known on the partition function or the correlation functions when the model has no magnetic field and is restricted to the (dual of the) disorder varieties. Of course, the low-temperature expansion obtained in this way up to order twelve will be checked by exhibiting exhaustively the corresponding diagrams. Such a low-temperature expansion for the checkerboard Potts model in a magnetic field will be used in § 3 to analyse systematically the simplifications occurring on the magnetisation, susceptibility (and higher field derivatives) when the model without magnetic field is restricted to the dual of the disorder variety. It will be conjectured that the magnetisation actually becomes equal to one on the dual of the disorder variety. In the vicinity of these varieties the expansion will also be seen to simplify drastically.

2. Low-temperature expansion of the q-state checkerboard Potts model in a magnetic field

The partition function per site Z of the q-state checkerboard scalar Potts model in a magnetic field is given by

$$Z^N(a,b,c,d;h)=\sum_{\{\sigma\}}\prod_{\langle ij\rangle} a^{\delta\sigma_i,\sigma_j}\prod_{\langle jk\rangle} b^{\delta\sigma_j,\sigma_k}\prod_{\langle kl\rangle} c^{\delta\sigma_k,\sigma_l}\prod_{\langle li\rangle} d^{\delta\sigma_l,\sigma_i}\prod_{m} h^{\delta\sigma_m,0}. \tag{1}$$

Here a, b, c and d denote the four parameters: $a=\mathrm{e}^{K_1}$, $b=\mathrm{e}^{K_2}$, $c=\mathrm{e}^{K_3}$ and $d=\mathrm{e}^{K_4}$ where K_i $(i=1,2,3,4)$ are the four coupling constants of the model (see figure 1) and $h=\mathrm{e}^H$ where H is the magnetic field. Each of the N spins σ_i of the lattice belongs to $\mathbb{Z}_q$ and the ordered sequences $\langle ij\rangle$, $\langle jk\rangle$, $\langle kl\rangle$ and $\langle li\rangle$ denote the edges of the N plaquettes.

The low-temperature expansion of the partition function per site gives the expression of the (low-temperature) normalised partition function per site defined by

$$Z(a,b,c,d;h)=(abcdh^2)^{1/2}\Lambda(a,b,c,d;h). \tag{2}$$

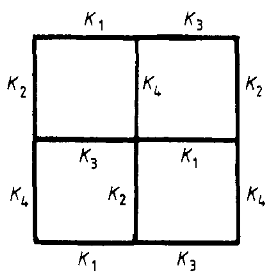


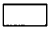
Figure 1. The elementary cell for the checkerboard Potts model.

2.1. *The low-temperature expansion*

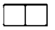
The parameters of the low-temperature expansion of the checkerboard Potts model will be denoted $A=1/a$, $B=1/b$, $C=1/c$, $D=1/d$ and $z=1/h$. The order ϑ of each diagram appearing in the expansion is the number of bonds of the diagram. The first terms of $\ln \Lambda$ are given by

$\ln \Lambda(A, B, C, D; z) =$

$$(q-1)zABCD \qquad (\vartheta = 4)$$



$$+\tfrac{1}{2}(q-1)z^2(A^2B^2C^2+A^2B^2D^2+A^2C^2D^2+B^2C^2D^2) \qquad (\vartheta = 6)$$



$$+\tfrac{1}{2}(q-1)(q-2)z^2(AB^2C^2D^2+A^2BC^2D^2+A^2B^2CD^2+A^2B^2C^2D) \qquad (\vartheta = 7)$$

$$+\dots \qquad (3)$$

This expansion is given exhaustively up to order twelve ($\vartheta = 12$) in the appendix and the corresponding diagrams are detailed. Note that the exponent of z in this expansion is simply given by the total area of the corresponding diagram and that for convenience the diagrams are represented as high-temperature diagrams instead of low-temperature ones.

One remarks that this expansion is not only invariant under the group of symmetry of the square C_{4v} as it should be, but is also invariant under the full group of permutation of the four coupling constants S_4. This confirms the existence of this unexpected hidden symmetry on the checkerboard Potts model (Maillard and Rammal 1985).

One can easily verify that this expansion is in agreement with the different expansions known in the literature on various lattices. For instance, in the limit $A = B = C = D$ one recovers, up to order twelve, the low-temperature expansion of the isotropic q-state Potts model on a square lattice in a magnetic field given by Straley and Fisher (1973). For $q = 2$, in the limit $A = B = C$, $D = 0$, one recovers the low-temperature expansions up to order twelve for the isotropic triangular Ising model in a magnetic field (Sykes *et al* 1973). For $q = 2$, in the limit $A = B = C$, $D = 1$, one recovers, up to order seven, the low-temperature expansion for the isotropic honeycomb lattice in a magnetic field for the ($q = 2$) Ising model (Sykes *et al* 1973). Note that diagrams of order greater than twelve contribute at order lower than twelve in the limit $D = 1$. That is why the expansion for the honeycomb lattice deduced from our expansion works only up to order seven.

The low-temperature expansion of the magnetisation for $q = 3$ on the triangular isotropic lattice (Enting 1980) is also in agreement with the expansion of the appendix.

An interesting simple limit of this expansion in a magnetic field is obtained by requiring that two of the four coupling constants of the model vanish. In this limit the partition function per site of the checkerboard model reduces to that of a one-dimensional Potts model in a magnetic field that can be calculated very easily. One obtains

$$\ln \Lambda(A, B, 1, 1; z) = \ln \lambda(A, B; z) \qquad (4)$$

where λ is a solution of a second-order algebraic equation:

$$0 = \lambda^2 - \lambda\{1 + 2(q-1)ABz + [1+(q-2)A][1+(q-2)B]z^2\} + (1-A)[1+(q-1)A](1-B)[1+(q-1)B]z^2. \qquad (5)$$

The expansion of $\ln \lambda$ for $A = B$ gives

$$\ln \lambda = \frac{2(q-1)}{1-z} zA^2 + \frac{2(q-1)(q-2)^2}{(1-z)^2} z^2A^3 + \frac{[(2q^3-9q^2+14q-7)z^3 + (-3q^2+6q-3)z^2]}{(1-z)^3} A^4 + \dots$$

One verifies that the expansion of (4) up to order four in (A, B) and order three in z is in agreement with the limit of expansion (3) for $C = D = 1$ and $A = B$.

Other limits on the model give only rather weak constraints on the expansion. For instance, the low-temperature expansion has to satisfy the following equation corresponding to the two different limits of the anisotropic square model from the checkerboard one:

$$\ln \Lambda(A, B, A, B; z^2) = 2 \ln \Lambda(A, B, 0, 1; z). \tag{6}$$

2.2. Agreement with (the dual of) the disorder solutions

In the absence of the magnetic field the model is self-dual. Therefore the low-temperature expansion of $\ln \Lambda$ and the high-temperature expansion of the corresponding normalised partition function are the same. The high-temperature expansions of the checkerboard Potts model without magnetic field can be seen to agree with the exact results for the partition function (Baxter 1984, Jaekel and Maillard 1985) and the nearest-neighbour correlation functions (Dhar and Maillard 1985) when the model is restricted to the disorder varieties. This means that the low-temperature expansions also simplify drastically on the dual of the disorder varieties.

The results are actually as follows. Restricted to the dual of the disorder variety

$$D + ABC + (q-2)ABCD = 0 \tag{7}$$

$\ln \Lambda$ is equal to

$$\ln \Lambda(A, B, C, D; 1) = \tfrac{1}{2} \ln[1 + (q-1)ABCD] = \tfrac{1}{2} \ln\left(1 - \frac{(q-1)A^2B^2C^2}{1 + (q-2)ABC}\right) \tag{8}$$

and

$$\frac{\partial}{\partial A} \ln \Lambda \bigg|_{(7)} = 0 = \frac{\partial}{\partial B} \ln \Lambda \bigg|_{(7)} = \frac{\partial}{\partial C} \ln \Lambda \bigg|_{(7)} \tag{9}$$

and also

$$\begin{aligned} \frac{\partial}{\partial D} \ln \Lambda \bigg|_{(7)} &= \frac{-(q-1)D[1 + (q-2)D/2]}{[1 + (q-1)D][1 + (q-2)D](1 - D)} \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}D} \ln\left(1 - \frac{(q-1)D^2}{1 + (q-2)D}\right). \end{aligned} \tag{10}$$

These varieties do not lie in the physical domain. For that reason (8)-(10) have to be seen as *formal* constraints on the analytic continuation (to non-physical domain of the parameters) of the low-temperature expansion of the partition function per site.

The two previous partial derivatives are related to nearest-neighbour correlation functions that can be calculated when restricted to the disorder variety (or formally to its dual). Because of the S_4 symmetry of permutation of the coupling constants there is no difference between the partial derivatives with respect to A, B or C.

From a diagrammatic point of view it is possible to shed some light on the simplification of the expansion. For any diagram each bond D can (in general) be replaced by three bonds or four bonds (see figure 2).

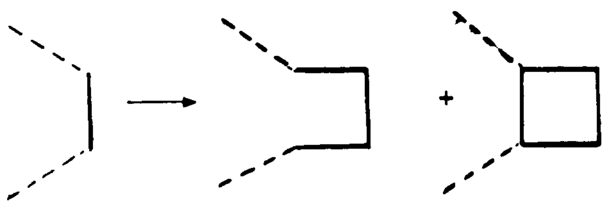


Figure 2. A generic diagram and its two diagrams associated under the disorder (dual of the disorder) condition.

Thus most of the diagrams of order ϑ can be associated with two other diagrams of order $\vartheta+2$ and $\vartheta+3$, such that the sum of these three contributions vanishes when condition (7) is satisfied.

These three equations are constraining enough to try to obtain the lowest orders of the low-temperature expansion of $\ln\Lambda$ order by order. Let us consider the expansion of $\ln\Lambda$ up to order eight. In this expansion one has two terms: $\frac{1}{2}(q-1)A^2B^2D^2$ and $(q-1)A^3B^3CD$. When one restricts the parameter space to the dual of the disorder variety (7) $\ln\Lambda$ must be equal to the expansion of (8). For equation (8) (and also (9) and (10)) to be verified requires the existence of a certain number of counterterms of higher orders. Actually at this order the counterterm is $\frac{1}{2}(q-1)A^4B^4C^2$. Unfortunately the counterterms are not always unique as in the previous example. For instance a term such as $A^\alpha B^\beta C^\gamma$ can come from $A^{\alpha-\delta}B^{\beta-\delta}C^{\gamma-\delta}D^\delta$ for different values of δ. Most of the time this ambiguity can be avoided by using some complementary information (known isotropic expansions, quick examination of some classes of diagrams, etc).

Although not completely systematic, this approach is powerful enough to obtain many of the contributions of the checkerboard low-temperature series. Indeed we have used this approach for computing most of the diagrams up to order twelve. Moreover, one finds the surprising result that, although our expansion is truncated at order twelve, it actually satisfies equations (8)-(10) respectively up to order thirteen, fourteen and fourteen in A, B, C. This means that no new constraints and new counterterms can be obtained at order thirteen from disorder solutions.

3. Analysis of the expansion of the checkerboard model in a magnetic field

3.1. *Analysis of the expansion on the dual of the disorder variety*

The low-temperature expansions of the magnetisation, the susceptibility (and higher field derivatives) at zero magnetic field can be obtained straightforwardly up to order twelve from expansion (3) by differentiation with respect to the variable z at $z=1$. Let M denote the spontaneous magnetisation of the checkerboard Potts model. One has

$$\begin{aligned}\left(\frac{q-1}{q}\right)(1-M) &= z\frac{\partial}{\partial z}\ln\Lambda(A, B, C, D; z)\bigg|_{z=1}\\ &= (q-1)ABCD + (q-1)(B^2C^2D^2 + S_4 \text{ symmetry terms})\\ &\quad + (q-1)(q-2)(AB^2C^2D^2 + \ldots). \qquad (11)\end{aligned}$$

In the case of the checkerboard Ising model the spontaneous magnetisation has been calculated exactly (Syozi and Naya 1960a, b) and the result is quite simply expressed in terms of the modulus of the elliptic functions that occur in the model

$$M=(1-k^2)^{1/8}. \tag{12}$$

It has also been remarked that this modulus trivialises on the disorder condition and also on its dual (Jaekel and Maillard 1984). Actually the modulus k^2 vanishes when condition (7), for $q=2$, is satisfied:

$$D+ABC=0 \Rightarrow k^2=0 \Rightarrow M=1. \tag{13}$$

Generalisation of this result for arbitrary q is a quite natural question. Indeed one can verify that, when condition (7) is satisfied, expansion (11) vanishes up to order twelve. Therefore we conjecture the following result on the low-temperature expansion of the spontaneous magnetisation:

$$D+ABC+(q-2)ABCD=0 \Rightarrow M=1. \tag{14}$$

Expansions of higher derivatives with respect to the magnetic field have also been calculated and drastic simplifications of these expansions have been obtained when one assumes (7):

$$\left(z\frac{\partial}{\partial z}\right)^2 \ln \Lambda\bigg|_{z=1,(7)} = (q-1)(ABC)^2-(q-1)(q-2)(ABC)^3$$
$$+(q-1)(A^2B^4C^4+S_3\,\mathrm{sym})+(q-1)(q-2)(A^3B^4C^4+\dots)$$
$$+(q-1)(q-2)(q-4)(ABC)^4+\dots \tag{15}$$

$$\left(z\frac{\partial}{\partial z}\right)^3 \ln \Lambda\bigg|_{z=1,(7)} = 3(q-1)A^2B^2C^2-3(q-1)(q-2)A^3B^3C^3$$
$$+9(q-1)(A^2B^4C^4+S_3\,\mathrm{sym})+9(q-1)(q-2)(A^3B^4C^4+\dots)$$
$$+3(q-1)(q^2-12q+22)A^4B^4C^4+\dots. \tag{16}$$

Equation (14) is not easy to prove rigorously, particularly because the algebraic variety (7) lies in a non-physical region of the parameter space. However, one can heuristically understand this conjecture. This amounts to performing a duality transformation on the model and, using a decimation procedure described elsewhere (Jaekel and Maillard 1985, Dhar and Maillard 1985), to 'eat' the dual lattice from the top and the bottom (assuming that this dual lattice satisfies appropriate boundary conditions) so that the magnetisation reduces to a calculation on a finite lattice.

The same method can in principle be used to calculate exactly the susceptibility and higher derivatives with respect to z of $\ln \Lambda(z)$ restricted to (7). This decimation suggets that some of these expressions could be rational expressions on (7).

3.2. Analysis of the expansion without magnetic field in the vicinity of the dual of the disorder variety

Let us first briefly recall the analysis of the vicinity of the disorder variety of the anisotropic triangular Ising model (Georges *et al* 1986a, b). The expansion of the high-temperature normalised partition function per site of the triangular Ising model without magnetic field is at first order in the vicinity of the disorder variety $t_3+t_1t_2=0$ ($t_i=\tanh K_i$ are the high-temperature (HT) variables for the model) as follows:

$$\ln \Lambda_{\mathrm{HT}}(t_1,t_2,t_3)=\tfrac{1}{2}\ln(1+t_1t_2t_3)+t_1t_2(t_3+t_1t_2)(1+t_1t_2t_3)^{-1}+\dots. \tag{17}$$

Remarkably all the coefficients of higher order in $t_3+t_1t_2$ are algebraic expressions of t_1 and t_2.

These expansions can simultaneously be seen as expansions in the vicinity of the dual of this disorder variety $(C+AB=0)$ of the low-temperature expansion for the normalised partition function of the honeycomb lattice:

$$\ln \Lambda(A, B, C, 1; 1)|_{q=2} = \tfrac{1}{2}\ln(1+ABC) + AB(C+AB)(1+ABC)^{-1} + \ldots . \tag{18}$$

In the case of the q-state checkerboard Potts model, the analysis of the vicinity of the dual of the disorder variety (7) can be performed using expansion (3) for $z=1$. This amounts to rewriting expansion (3) as

$$\ln \Lambda(A, B, C, D; 1) = \tfrac{1}{2}\ln[1+(q-1)ABCD] + \sum_{l=1}^{\infty} \alpha_l[D+ABC+(q-2)ABCD]^l. \tag{19}$$

This analysis is of course at the same time the analysis of the vicinity of the disorder varieties when A, B, C, D are seen as high-temperature variables. The first-order term is known exactly. α_1 is nothing but the nearest-neighbour correlation function (10):

$$\alpha_1 = \frac{-(q-1)[1+\frac{1}{2}(q-2)D]D}{[1+(q-1)D][1+(q-2)D](1-D)}$$
$$= (q-1)ABC[1+\tfrac{1}{2}(q-2)ABC+(q-1)^2A^2B^2C^2+\ldots]. \tag{20}$$

The following expansion has been obtained for α_2:

$$\begin{aligned}\alpha_2 = {} & (q-1)(q-2)(q-3)(A^4B^4C^2+S_3\,\mathrm{sym})+(q-2)(q-1)(A^4B^4C+\ldots)\\ & +\tfrac{1}{2}(q-1)(A^4B^4+\ldots)-3(q-1)^2(q-2)A^3B^3C^3\\ & +(q-1)^2(q-2)(A^3B^3C^2+\ldots)\\ & +(q-1)(q-2)(A^3B^3C+\ldots)-\tfrac{1}{2}(q-1)(5q-7)A^2B^2C^2\\ & +\tfrac{1}{2}(q-2)(q-1)(A^2B^2C+\ldots)+\tfrac{1}{2}(q-1)(A^2B^2+\ldots).\end{aligned} \tag{21}$$

These expansions are simple. The α_i can be obtained exactly for $q=2$. It would be interesting to have their exact expressions for arbitrary q.

With the same motivation, it is interesting to consider the expansion of the magnetisation in the vicinity of the algebraic variety (7). If the conjecture (14) is true then one has

$$M = 1 + \sum_{l=1}^{\infty} \beta_l[D+ABC+(q-2)ABCD]^l. \tag{22}$$

We have obtained the following simple expansion for β_1:

$$\begin{aligned}\beta_1 = {} & (q-1)ABC[1+(A^2B^2C+\ldots)-2(q-2)A^2B^2C^2+(A^2B^2C^4+\ldots)\\ & +2(q-2)(A^2B^3C^3+\ldots)+\ldots].\end{aligned} \tag{23}$$

Let us come back to the honeycomb (triangular) Ising model. In expansion (18) the variety $C+AB=0$ is singled out. Similar results for the vicinity of the two other varieties $A+BC=0$ and $B+AC=0$ can be synthesised in the following way:

$$\ln \Lambda(A, B, C, 1; 1) = \tfrac{1}{2}\ln(1+ABC) + \frac{(A+BC)(B+AC)(C+AB)}{(1-A^2)(1-B^2)(1-C^2)} + \ldots . \tag{24}$$

One recovers expansion (18) for the vicinity of the variety $C+AB=0$ from expansion (24). The symmetry of permutation of the three variables A, B and C is explicit on expansion (24).

One would like to generalise this simultaneous expansion in the vicinity of all the disorder varieties to the case of the q-state checkerboard Potts model, i.e. to find a function $F(A, B, C, D)$ such that

$$\ln \Lambda(A, B, C, D, 1) = \tfrac{1}{2}\ln[1+(q-1)ABCD] + F(A, B, C, D)\prod(D+ABC+(q-2)ABCD)+\ldots \tag{25}$$

where $\Pi(\ldots)$ denotes the product over the four duals of the disorder varieties. The expansion of F is actually very simple:

$$F = \tfrac{1}{2}(q-1)[1+(A^2B^2+\ldots)+(q-2)(AB^2C^2+\ldots)+(q-1)(A^2B^2C^2+\ldots) +(A^4B^4+\ldots)-(q-2)^2(A^3B^3C^2+\ldots)]. \tag{26}$$

One remarks that for $q=2$ (26) is in agreement with equation (24). It can be interesting to compare expansion (3), given in the appendix up to order twelve, and expansion (26). The four coefficients of F are sufficient to recover almost two thirds of the coefficients of expression (3).

4. A comment on the susceptibility of the anisotropic Ising model

Disorder solutions (respectively their dual) have been seen to play an important role in understanding and analysing the high (respectively low) temperature expansions on the checkerboard Potts model. One would like to understand more clearly the limit of such an approach. For instance, let us suppose that for a certain quantity one can exhibit a closed expression in agreement with the disorder solutions with all the limits that can be calculated exactly (one-dimensional limits, etc), with all the symmetries of the model (symmetries of permutation of the coupling constants, inversion relation, duality, etc) and, of course, in agreement up to a certain order, with the anisotropic expansions available on the model. If such a closed expression is not the exact one, is it possible to characterise the discrepancy with the exact value of this quantity? In particular one might wonder if such a closed expression could present the same singularity as the exact one. To shed some light on this question let us consider the example of the susceptibility of the anisotropic triangular Ising model and its anisotropic high-temperature expansion.

Using symmetry properties of the Ising model (star-triangle relation, duality transformation, symmetry of permutation of the coupling constants, etc), the one-dimensional limits and the anisotropic high-temperature expansions, Syozi and Naya (1960a) proposed a closed expression for the susceptibility of the triangular Ising model:

$$\chi = \frac{(1-t_1^2)(1-t_2^2)(1-t_3^2)}{(1-t_1-t_2-t_3-t_1t_2-t_2t_3-t_1t_3+t_1t_2t_3)^2} M^{*2} \tag{27}$$

where

$$M^* = \left(1 - \frac{16(1+t_1t_2t_3)(t_1+t_2t_3)(t_2+t_1t_3)(t_3+t_1t_2)}{(1-t_1^2)^2(1-t_2^2)^2(1-t_3^2)^2}\right)^{1/8}. \tag{28}$$

Note that M^* is nothing else but the magnetisation of the triangular Ising model for which the low-temperature variables have been replaced by the high-temperature ones.

Actually expression (27) reduces to the exact result of Dhar and Maillard (1985) for the susceptibility restricted to the disorder condition:

$$t_3+t_1t_2=0 \tag{29}$$

$$\chi=\frac{(1+t_1)(1+t_2)(1+t_1t_2)}{(1-t_1)(1-t_2)(1-t_1t_2)}. \tag{30}$$

The closed expression (27) is singular on the critical variety of the model and has actually the correct critical exponent $\gamma=\frac{7}{4}$. Equation (27) is obviously related to the Rushbrook (1963) identity on the exponents in the Ising limit ($\alpha=0$):

$$-\gamma=2\beta-2+\alpha. \tag{31}$$

This expression satisfies the inversion relation (Jaekel and Maillard 1985)

$$\chi(t_1,t_2,t_3)+\chi(-t_1,-t_2,1/t_3)=0. \tag{32}$$

Moreover, one can compare, in the anisotropic square limit ($t_3=0$), the expansion of (27) in the variables t_2 with the exact resummed high-temperature expansion (well suited for the analysis of the inversion relation) for the susceptibility of the Ising model. The expansion of (27) is:

$$\begin{aligned}\chi=&\frac{1+t_1}{1-t_1}+2t_2\left(\frac{1+t_1}{1-t_1}\right)^2+t_2^2\frac{1+6t_1+8t_1^2+6t_1^3+t_1^4}{(1-t_1)^3(1+t_1)}+t_2^3\frac{1+8t_1+10t_1^2+8t_1^3+t_1^4}{(1-t_1)^4}\\&+2t_2^4\frac{t_1^8+14t_1^7+56t_1^6+122t_1^5+146t_1^4+122t_1^3+56t_1^6+14t_1+1}{(1-t_1)^5(1+t_1)^3}\\&+2t_2^5\frac{t_1^8+16t_1^7+64t_1^6+144t_1^5+166t_1^4+144t_1^5+64t_1^2+16t_1+1}{(1-t_1)^6(1+t_1)^2}+\dots\end{aligned} \tag{33}$$

All the terms of (33) of order less than four in t_2 are in agreement with the exact resummed high-temperature expansion for χ (Jaekel and Maillard 1985, Hansel *et al* 1987). The coefficient of t_2^4 is not the correct one as can be shown on its expansion in powers of t_1. However, it is very close to the exact result. Indeed, it leads to an exact coefficient for $t_1^4t_2^n$ $(0\leq n\leq 4)$ while it gives 14 416 $t_1^4t_2^7$ instead of 14 424 $t_1^4t_2^7$ for the exact high-temperature expansion up to order eleven (Oitmaa 1987). Surprisingly enough the coefficient of t_2^5 in (31) is exact (Hansel *et al* 1987).

Therefore this closed expression is not the exact expression for χ: the expansions of both expressions are different at order six in t_1, t_2, t_3. For simplicity let us consider the isotropic limit of the model. It is illuminating to compare the expansion of (27) with the exact isotropic high-temperature expansion for χ which is known up to order sixteen (Sykes *et al* 1972). They are obviously not very different: for instance the coefficient of order sixteen is 10 961 531 202 instead of the exact coefficient 10 969 820 358. The relative error is 7×10^{-5}. Let us remark after Syozi and Naya (1960a, b) that the amplitude of the singularity has very good agreement with the amplitude deduced from expansions ($A_+=0.9235$ instead of $0.924\,21\pm0.000\,03$ (Gaunt and Guttmann 1974, Guttmann 1976)), but it is definitely different from the exact amplitude.

Coming back to the case of the anisotropic square Ising model it is possible to analyse more precisely the discrepancy between Syozi and Naya's closed-form expression and the exact expression for the susceptibility. Actually Wu *et al* (1976) have obtained explicit analytic results for the zero-field susceptibility in the scaling limit. Above the critical temperature equation (7.45*a*) of Wu *et al* (1976) shows that the expression of Syozi and Naya is actually the full one-particle-excitation part of the susceptibility (ignoring three-particle, five-particle, etc, contributions (Tracy and McCoy 1975)).

This example shows that it is possible to exhibit closed expressions in agreement with the symmetries of the model (inversion relation, permutation of the coupling constants, etc) but also in agreement with the exact expressions on very different domains of the parameter space: high temperature, critical variety, dual of the disorder variety, one-dimensional limits, etc. When such a closed expression can be found it has been seen that, from many points of view, it can hardly be distinguished from the exact one. It would be interesting to analyse more precisely the discrepancy between such closed expressions and the exact one (many-particle contributions) and of course to apply these ideas to models that are not exactly solvable.

The triangular Potts model is obviously a good candidate to generalise these ideas. It is not an exactly solvable model (except at criticality) but it presents many remarkable features; the critical variety and the critical exponents of the model are known exactly (see Wu (1982) and Rammal and Maillard (1983) for instance) and there exist disorder solutions and inversion relation symmetries on these models. Of course one cannot expect a simple closed algebraic expression such as (28) for the magnetisation M^* but one can expect that the susceptibility χ could satisfy, with the magnetisation M^* and a quantity related to the thermal exponent α, an equation of state associated with the Rushbrook identity. This would sum up the ideas we have tried to promote in this paper; the expansions simplify drastically on the dual of the disorder solutions and it can be interesting to introduce a variable that synthesises these simplifications and all the exact properties of the model. Actually the analytic continuation of the low-temperature expansion of the magnetisation formally reduces to a constant on the dual of the disorder varieties (conjecture (14)), it vanishes on the critical variety, it is invariant under the symmetry of permutation of the coupling constants and it is also invariant under the inversion relation for the model (these symmetries generate an infinite discrete group, see Jaekel and Maillard (1984)). The situation we describe amounts to saying that 'most' of the complexity of the problem would be taken into account by introducing an appropriate variable, for instance the order parameter M^*.

Acknowledgments

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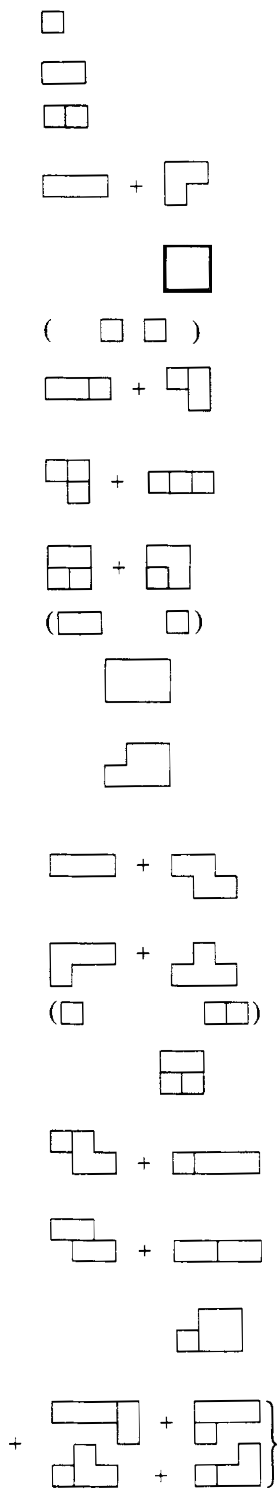
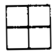
Appendix

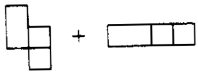
The low-temperature expansion of the normalised partition function per site for the checkerboard Potts model in a magnetic field up to order twelve in the low-temperature variables A, B, C and D is given by



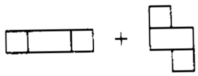
$$
\begin{aligned}
&\ln \Lambda(A, B, C, D; z) = \\
&(q-1)zABCD && (\vartheta = 4)\\
&+\tfrac{1}{2}(q-1)z^2(B^2C^2D^2+\ldots) && (\vartheta = 6)\\
&+\tfrac{1}{2}(q-1)(q-2)z^2(AB^2C^2D^2+\ldots) && (\vartheta = 7)\\
&+(q-1)z^3(A^3B^3CD+\ldots) && (\vartheta = 8)\\
&+(q-1)z^4A^2B^2C^2D^2\\
&-\tfrac{5}{2}(q-1)^2z^2A^2B^2C^2D^2\\
&+(q-1)(q-2)z^3(A^3C^3BD^2+\ldots) && (\vartheta = 9)\\
&+(q-1)(q-2)^2z^3(A^3C^3B^2D^2+\ldots) && (\vartheta = 10)\\
&+(q-1)(q-2)z^4(A^2B^2C^3D^3+\ldots)\\
&-4(q-1)^2z^3(A^3B^3C^3D+\ldots)\\
&+\tfrac{1}{2}(q-1)z^6(A^4B^2C^2D^2+\ldots)\\
&+2(q-1)z^5(A^3B^3C^3D+\ldots)\\
&+\tfrac{1}{2}(q-1)z^4(A^4B^4C^2+\ldots)\\
&+3(q-1)z^4(A^4B^2C^2D^2+\ldots)\\
&-4(q-1)^2(q-2)z^3(A^2B^3C^3D^3+\ldots) && (\vartheta = 11)\\
&+(q-1)(q-2)(q-3)z^4(A^3B^3C^3D^2+\ldots)\\
&+(q-1)(q-2)z^4(A^4B^4C^2D+\ldots)\\
&+\tfrac{1}{2}(q-1)(q-2)z^4(A^4B^4C^3+\ldots)\\
&+(q-1)(q-2)z^5(A^3B^3C^3D^2+\ldots)\\
&+(q-1)(q-2)z^4(A^4B^3C^2D^2+\ldots)
\end{aligned}
$$



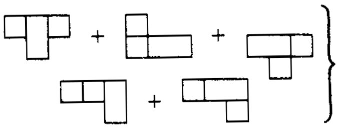
$$+(q-1)(q-2)(q^2-5q+7)z^4A^3B^3C^3D^3 \qquad (\vartheta=12)$$



$$+(q-1)(q-2)^2z^4(A^4B^4C^3D+\ldots)$$

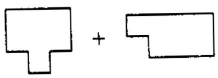


$$+(q-1)(q-2)^2z^4(A^4B^4C^2D^2+\ldots)$$

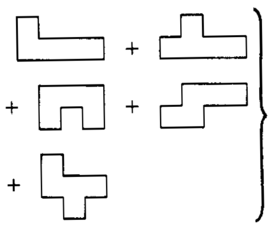


$$+3(q-1)(q-2)^2z^4(A^3B^3C^4D^2+\ldots)$$

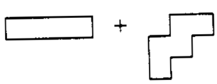
$$+\tfrac{31}{3}(q-1)^3z^3A^3B^3C^3D^3$$



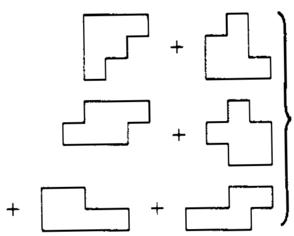
$$+(q-1)z^7(A^5B^3C^3D+\ldots)$$



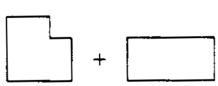
$$+3(q-1)z^5(A^5B^3C^3D+\ldots)$$



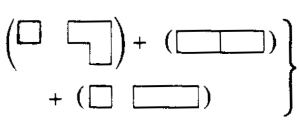
$$+(q-1)z^5(A^5B^5CD+\ldots)$$



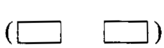
$$+(q-1)z^6(A^4B^4C^2D^2+\ldots)$$



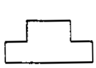
$$+(q-1)z^8(A^4B^4C^2D^2+\ldots)$$



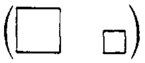
$$-13(q-1)^2z^4(A^4B^4C^2D^2+\ldots)$$



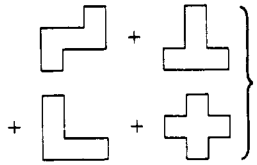
$$-\tfrac{7}{4}(q-1)^2z^4(A^4B^4C^4+\ldots)$$



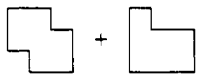
$$+(q-1)z^6(A^4B^4C^4+\ldots)$$



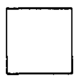
$$-12(q-1)^2z^5A^3B^3C^3D^3$$



$$+13(q-1)z^5A^3B^3C^3D^3$$



$$+10(q-1)z^7A^3B^3C^3D^3$$



$$+(q-1)z^9A^3B^3C^3D^3.$$

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